

Density and tails of unimodal convolution semigroups ^{*}

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Abstract

We give sharp bounds for the isotropic unimodal probability convolution semigroups when their Lévy-Khintchine exponent has weak local scaling at infinity of order strictly between 0 and 2.

1 Introduction

Estimating Markovian semigroups is important from the point of view of theory and applications because they describe evolution phenomena and underwrite various forms of calculus. Diffusion semigroups traditionally receive most attention [24] but considerable progress has also been made in studies of transition densities of rather general jump-type Markov processes. Such studies are usually based on assumptions concerning the profile of the jump or Lévy kernel (measure) at the diagonal (origin) and at infinity [11, 9]. As a rule the assumptions can be viewed as approximate or weak scaling conditions for the Lévy density, to which some structure conditions may be added, see [11, (1.9)-(1.14) and Theorem 1.2]. Typical results consist of sharp two-sided estimates of the heat kernel for small and/or large times.

Transition semigroups of Lévy processes allow for a deeper insight and direct approach from several directions thanks to their convolutional structure and the available Fourier techniques. For instance, the upper bounds for transition densities of isotropic Lévy densities with relatively fast decay at infinity are obtained in [20, 21] by using Fourier inversion, complex integration, saddle point approximation or the Davies' method. In this work we study one-dimensional distributions $p_t(dx)$ of rather general isotropic unimodal Lévy processes $X = (X_t, t \geq 0)$ in \mathbb{R}^d [26]. We focus on pure-jump isotropic unimodal Lévy processes. Thus, X is a càdlàg stochastic process with distribution \mathbb{P} , such that $X(0) = 0$ almost surely, the increments of X are independent with rotation invariant and radially nonincreasing density function $p_t(x)$ on $\mathbb{R}^d \setminus \{0\}$, and the following Lévy-Khintchine formula holds for $\xi \in \mathbb{R}^d$,

$$\mathbb{E}e^{i\langle \xi, X_t \rangle} = \int_{\mathbb{R}^d} e^{i\langle \xi, x \rangle} p_t(dx) = e^{-t\psi(\xi)}, \quad \text{where} \quad \psi(\xi) = \int_{\mathbb{R}^d} (1 - \cos \langle \xi, x \rangle) \nu(dx). \quad (1)$$

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Here and below ν is an isotropic unimodal *Lévy measure* and \mathbb{E} is the integration with respect to \mathbb{P} . Further notions and definitions are given in Sections 2 and 3 below. Put differently, we study the vaguely continuous spherically isotropic unimodal convolution semigroups $(p_t, t \geq 0)$ of probability measures on \mathbb{R}^d with purely nonlocal generators. (In this work we never use probabilistic techniques beyond the level of one-dimensional distributions of X .)

Our main result provide estimates for the tails of p_t and its density function $p_t(x)$, expressed in terms of the Lévy-Khintchine exponent ψ . We also use ψ to estimate the density function $\nu(x)$ of the Lévy measure ν . Since ψ is radially *almost increasing*, it is *comparable* with its radial nondecreasing majorant ψ^* , and as a rule we employ ψ^* in statements and proofs. The extensive use of ψ (ψ^*) rather than ν is a characteristic feature of our development and may be considered natural from the point of view of pseudo-differential operators and spectral theory [16]. As usual, the asymptotics of ψ at infinity translates into the asymptotics of ν and p_t at the origin. Our estimates may be summarized as follows,

$$p_t(x) \approx [\psi^-(1/t)]^d \wedge \frac{t\psi^*(|x|^{-1})}{|x|^d} \approx p_t(0) \wedge [t\nu(x)], \quad (2)$$

see Theorem 21, Corollary 23 and (24) for detailed statements. Here \approx means that both sides are *comparable* i.e. their ratio is bounded between two positive constants, ψ is assumed to satisfy the so-called *weak upper* and *lower scalings* of order strictly between 0 and 2, and ψ^- is the generalized inverse of ψ^* . The bound (2) holds locally in time and space, or even globally, if the scalings are global. We note that the corresponding estimates of ν , to wit,

$$\nu(x) \approx \frac{\psi^*(|x|^{-1})}{|x|^d}, \quad (3)$$

are simply obtained as a consequence of (2) for small time, and not as an element of its proof (see Corollary 23). It is common for ν to share the asymptotics with p_t because $\nu = \lim_{t \rightarrow 0} p_t$, a vague limit in $\mathbb{R}^d \setminus \{0\}$. It is also a manifestation of the general rule mentioned above that $\psi^*(|x|^{-1})$ in our estimates reflect the properties of $p_t(x)$ and $\nu(x)$. The denominator, $|x|^d$, in the estimates comes from the homogeneity of the volume measure in \mathbb{R}^d (see the proof of Corollary 23) and $\psi^-(1/t)$ approximates $p_t(0)$ as follows from a change of variables in Fourier inversion (cf. 26, Lemma 16 and Lemma 17). All these reasons make the above bounds most natural. Therefore below we shall address (two-sided and one-sided) estimates similar to (2) and (3) as *common* bounds. We note that the *common* upper bounds,

$$p_t(x) \leq C \frac{t\psi^*(1/|x|)}{|x|^d}, \quad \nu(x) \leq C \frac{\psi^*(1/|x|)}{|x|^d}, \quad x \in \mathbb{R}^d \setminus \{0\}, \quad t > 0,$$

hold with a constant depending only on the dimension for *all* isotropic unimodal Lévy semigroups. This is proved in Corollary 7. The lower *common* bounds hold if and only if ψ has (the so-called weak) lower and upper scalings of order strictly between 0 and 2. Indeed, in Theorem 26 we show that for unimodal Lévy processes, the scaling of ψ (at infinity) is *equivalent* to the *common* bounds for the transition density and the Lévy measure (at the origin). In fact, already the lower bound $\nu(x) \geq c\psi^*(|x|^{-1})/|x|^d$ implies such scalings of ψ .

We thus cover all the cases of isotropic Lévy processes with scaling studied in literature, and upper bounds are provided for all isotropic unimodal Lévy processes, which (only) leaves open the problem of estimating the semigroup of (scaling) unimodal Lévy process with Lévy-Khintchine exponent slowly varying at infinity or regularly 2-varying at infinity [6, 22]. Here typical examples are the geometric stable processes and the variance gamma process—the Brownian

motion subordinated by an independent gamma subordinator. As we see from [6, Section 5.3.4], such processes require specialized approach and their transition density and Lévy-Khintchine exponent do not easily explain each other.

Bochner's procedure of subordination is strongly rooted in semigroup and operator theory, harmonic analysis and probability [32, 27, 26]. In the present setting it yields a wide array of asymptotics of ψ , p_t and ν , which explains intense current developments. In particular, *common* bounds were recently obtained for a class of subordinate Brownian motions, mainly for the *complete* subordinate Brownian motions defined by a delicate structure condition [26]. Highly sophisticated current techniques and results in this direction are presented in [18], see also [6, 26]. Our approach is, however, more general and synthetic; we demonstrate that the sources of the asymptotics of (unimodal) $p_t(x)$ are merely its radial monotonicity and the scalings of ψ , rather than further structure properties of ψ .

We illustrate our results with several classes of relevant examples. These include situations where former methods cannot be easily applied and the present method works well. In this connection we note that ψ is an integral quantity and may exhibit less variability than ν , in particular the scaling properties of ψ are more easily manageable. Many of our examples are in fact the subordinate Brownian motions, and then $\psi(\xi) = \phi(|\xi|^2)$, where ϕ is the Laplace exponent of the subordinator, i.e. a Bernstein function. There is by now an impressive pool of Bernstein functions studied in the literature, with distinct asymptotics at infinity. For instance the monograph [26] gives well over one hundred cases and classes of Bernstein functions in its closing list of examples. Many of these functions have the scaling properties used in our paper. This immediately yields sharp estimates of the Lévy measure and transition density of subordinate Brownian motions corresponding to such subordinators. In comparison, former methods require to first find estimates of the Lévy measure of the subordinator (this is where the completeness of the subordinator plays a role), then to estimate the Lévy density of the corresponding subordinate Brownian motion and then, finally, to estimate its semigroup [10, 18, 6].

Our results and arguments are purely real-analytic. We circumvent those additional steps and directly estimate the heat kernel by arguments not unrelated to integration by parts. In particular, for subordinate Brownian motions with scaling, we relax the usual completeness assumption on the subordinator. Noteworthy, while on one hand the completeness need not be used when scaling is present, on the other hand the complete subordinate Brownian motions exhibit all the types of asymptotics of ψ , p_t and ν of general unimodal Lévy processes with scaling. This is in particular manifested in Corollary 27.

We remark that analogues of the on-diagonal term $[\psi^-(1/t)]^d \approx p_t(0)$ in (2) are often obtained for more general Markov processes via Nash and Sobolev inequalities [3, 8, 2, 29, 21]. For our unimodal Lévy processes we instead use Fourier inversion and (weak) lower scaling, see Proposition 19. Also our approach to the off-diagonal term $t\psi^*(|x|^{-1})/|x|^d$ is very different and much simpler than the arguments leading to the upper bounds in the otherwise more general Davies' method [8], [12, Section 3]. Our *common* upper bounds are straightforward consequences of a specific quadratic parametrization of the tail function, which is crucial in applying the techniques of Laplace transform. The *common* lower bounds are harder, and they are intrinsically related to upper and lower scalings via certain differential inequalities in the proof of Theorem 26.

The (local or global) comparability of the *common* lower and upper bounds is a remarkable feature of the class of semigroups captured by Theorem 26. We expect further applications of the estimates. For instance, under (weak) global scalings we obtain important metric-type [17] global comparisons $p_{2t}(x) \approx p_t(x)$ and $p_t(2x) \approx p_t(x)$, given in Corollary 24 below. These should

matter in perturbation theory of Lévy generators and in nonlinear partial integro-differential equations. Since uniform estimates are important in some applications, in Corollary 24 and elsewhere in the paper the *comparability constants* are shown to depend in a rather explicit way on specific properties of the semigroups, chiefly on scaling. For instance for the isotropic α -stable Lévy semigroup in \mathbb{R}^d with $0 < \alpha < 2$ [6], we have $\psi(\xi) = |\xi|^\alpha$, and we arrive at

$$c_\alpha \left(t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}} \right) \leq p_t(x) \leq C_\alpha \left(t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}} \right), \quad (4)$$

with explicit constants given by (17), (26) and (29) below. Noteworthy, our (weak) scaling conditions imply majorization and minorization of ψ at infinity by power functions with exponents strictly between 0 and 2, but do not require its *comparability* with a power function, see examples in Section 4.1. Furthermore, the exponents $\underline{\alpha}$ and $\overline{\alpha}$ in the assumed scalings only affect the comparison constants in *common* bounds, but not the rate of asymptotic, which is solely determined by ψ .

For convolution semigroups of probability measures more general than unimodal, the structure of the support and the regularity of the Lévy measure plays a crucial role. In particular the directions which are not charged by the Lévy measure see in general lighter asymptotics of $p_t(dx)$ [13, 31]. In consequence, the estimates of severely anisotropic convolution semigroups require completely different assumptions, description and methods. Our experience indicates that ν surpasses ψ in such cases. Estimates and references to anisotropic ν with prescribed radial decay and rough spherical marginals may be found in [28] (see also [7, 31] for more details in the case of *homogeneous* anisotropic ν).

The structure of the paper is as follows. In Section 2 we discuss first consequences of isotropy and radial monotonicity. In particular we compare ψ with Pruitt-type function $h_1(r) = \int_{\mathbb{R}^d} (1 \wedge |x_1|^2/r^2) \nu(dx)$ and we estimate from above the tail function $f_t(\rho) = \mathbb{P}(|X_t| \geq \sqrt{\rho})$ by using Laplace transform and ψ . This and radial monotonicity quickly lead to upper bounds for $p_t(x)$ and $\nu(x)$. In Section 3 we discuss almost monotone and general weakly scaling functions. In Section 4 we specialize to scalings with lower and upper exponents $\underline{\alpha}, \overline{\alpha} \in (0, 2)$, and we give examples of ψ with such scaling. To obtain lower bounds for $p_t(x)$ and $\nu(x)$, in Lemma 13 we recall an observation due to M. Zähle, which is then used in Lemma 14 to reverse the comparison between ψ and the tail function f_t . The generalized inverse ψ^- plays a role through a change of variables in Fourier inversion formula for $p_t(0)$ in Lemma 16 and through equivalence relation defining "small times" (stated as Lemma 17). In Theorem 21 we combine all the threads to estimate p_t , as summarized by (2). In Corollary 23 we obtain (3) as a simple consequence of Theorem 21. To close the circle of ideas, in Theorem 26 we show the equivalence of the weak scalings with the *common* form of bounds for p_t and ν . In Proposition 28 we state a connection between ν and ψ for a class of *approximately* isotropic Lévy densities.

2 Unimodality

We shall often use the gamma and incomplete gamma functions:

$$\Gamma(\delta) = \int_0^\infty e^{-u} u^{\delta-1} du, \quad \gamma(\delta, t) = \int_0^t e^{-u} u^{\delta-1} du, \quad \Gamma(\delta, t) = \int_t^\infty e^{-u} u^{\delta-1} du, \quad \delta, t > 0.$$

Let \mathbb{R}^d be the Euclidean space of (arbitrary) dimension $d \in \mathbb{N}$. For $x \in \mathbb{R}^d$ and $r > 0$ we let $B(x, r) = \{y \in \mathbb{R}^d : |x - y| < r\}$, and $B_r = B(0, r)$. We denote by $\omega_d = 2\pi^{n/2}/\Gamma(n/2)$ the surface measure of the unit sphere in \mathbb{R}^d . All sets, functions and measures considered below are

(assumed) Borel. A (Borel) measure on \mathbb{R}^d is called isotropic unimodal, in short: unimodal, if on $\mathbb{R}^d \setminus \{0\}$ it is absolutely continuous with respect to the Lebesgue measure, and has a finite *radial nonincreasing* density function. Such measures may have an atom at the origin: they are of the form $a\delta_0(dx) + f(x)dx$, where $a \geq 0$, δ_0 is the Dirac measure,

$$f(x) = \int_0^\infty \mathbf{1}_{B_r}(x) \mu(dr) = \mu((|x|, \infty)) \quad (a.e.),$$

and $\mu((\varepsilon, \infty)) < \infty$ for all $\varepsilon > 0$. A Lévy process $X = (X_t, t \geq 0)$, is called isotropic unimodal (in short, unimodal) if all of its one-dimensional distributions (transition densities) $p_t(dx)$ are such. Recall that Lévy measure is any measure concentrated on $\mathbb{R}^d \setminus \{0\}$ such that

$$\int_{\mathbb{R}^d} (|x|^2 \wedge 1) \nu(dx) < \infty. \quad (5)$$

Unimodal pure-jump Lévy processes are characterized in [30] by unimodal Lévy measures $\nu(dx) = \nu(x)dx = \nu(|x|)dx$.

Unless explicitly stated otherwise, in what follows we assume that X is a pure-jump unimodal Lévy process in \mathbb{R}^d with (unimodal) *nonzero* Lévy measure (density) ν .

Each measure p_t is the weak limit of

$$p_t^\varepsilon = e^{-t\nu_\varepsilon(\mathbb{R}^d)} \sum_{n=0}^{\infty} \frac{1}{n!} \nu_\varepsilon^{*n},$$

where $\varepsilon \rightarrow 0^+$ and ν_ε^{*n} are convolution powers of measures $\nu_\varepsilon(A) = \nu(A \setminus B_\varepsilon)$, see, e.g., [6, Section 1.1.2]. Each p_t has a radial nonincreasing density function $p_t(x)$ on $\mathbb{R}^d \setminus \{0\}$ and atom $\exp[-t\nu(\mathbb{R}^d)]$ at the origin if $\nu(\mathbb{R}^d) < \infty$ (no atom if $t\nu$ is infinite).

For $r > 0$ we define after [23],

$$h(r) = \int_{\mathbb{R}^d} \left(\frac{|x|^2}{r^2} \wedge 1 \right), \quad L(r) = \nu(B_r^c). \quad (6)$$

Clearly, $0 \leq L(r) < h(r) < \infty$, L is nonincreasing and h is decreasing. The strict monotonicity and positivity of h follows since $\nu \neq 0$ is nonincreasing, hence positive near the origin.

The first coordinate process X_t^1 of X_t is unimodal in \mathbb{R} . The corresponding quantities $L_1(r)$ and $h_1(r)$ are given by the (pushforward) Lévy measure $\nu_1 = \nu \circ \pi_1^{-1}$, where π_1 is the projection: $\mathbb{R}^d \ni x = (x_1, \dots, x_d) \mapsto x_1$, see [25, Proposition 11.10]. With a typical abuse of notation we let $\nu_1(y)$ denote the (symmetric and nonincreasing on $(0, \infty)$) density function of ν_1 :

$$\nu_1(y) = \int_{\mathbb{R}^{d-1}} \nu(\sqrt{y^2 + |z|^2}) dz, \quad y \in \mathbb{R} \setminus \{0\}.$$

Thus,

$$h_1(r) = \int_{\mathbb{R}} \left(\frac{y^2}{r^2} \wedge 1 \right) \nu_1(dy) = \int_{\mathbb{R}^d} \left(\frac{|x_1|^2}{r^2} \wedge 1 \right) \nu(dx), \quad r > 0.$$

Therefore,

$$h_1(r) \leq h(r) \leq h_1(r)d, \quad r > 0. \quad (7)$$

In fact, (7) is valid more generally: for all rotation invariant Lévy measures.

Remark 1. $h(r)$ and $h_1(r)$ are decreasing, while $r^2h(r)$ and $r^2h_1(r)$ are nondecreasing.

Since ψ is a radial function, we shall often write $\psi(u) = \psi(\xi)$, where $\xi \in \mathbb{R}^d$ and $u = |\xi| \geq 0$. We obtain the same function for X_t^1 . Clearly, $\psi(0) = 0$ and, as before for h , $\psi(u) > 0$ for $u > 0$.

We now show how to use h_1 to estimate the Lévy-Khintchine exponent ψ of X .

Lemma 1.

$$\frac{2}{\pi^2} h_1\left(\frac{1}{u}\right) \leq \psi(u) \leq 2h_1\left(\frac{1}{u}\right), \quad u > 0. \quad (8)$$

Proof. For $t \geq 0$ we define $\kappa(t) = \int_0^t (1 - \cos r) dr$, and we claim that

$$\frac{2}{\pi^2} \int_0^t (r^2 \wedge 1) dr \leq \kappa(t) \leq 2 \int_0^t (r^2 \wedge 1) dr, \quad t \geq 0. \quad (9)$$

Indeed, $1 - \cos r = 2 \sin^2(r/2) \leq 2(r^2 \wedge 1)$, which gives the upper bound. If $0 \leq r \leq t \leq \pi$, then $1 - \cos r = 2 \sin^2(r/2) \geq 2r^2/\pi^2$ hence, $\kappa(t) \geq 2\pi^{-2} \int_0^t (r^2 \wedge 1) dr$, and if $t > \pi$, then

$$\int_{\pi/2}^t (1 - \cos r) dr = (t - \pi/2) + (1 - \sin t) \geq \int_{\pi/2}^t dr,$$

which yields (9). (The constants in (9) may be improved.) We define an auxiliary measure μ on $(0, \infty)$ by letting

$$\mu((y, \infty)) = \nu_1(y) \quad \text{for a.e. } y > 0.$$

Let $u > 0$. By a change of variables and Fubini-Tonelli,

$$\begin{aligned} \frac{1}{2} h_1\left(\frac{1}{u}\right) &= \int_0^\infty [(u^2 y^2) \wedge 1] \nu_1(y) dy = \int_0^\infty [(u^2 y^2) \wedge 1] \int_{(y, \infty)} \mu(dt) dy \\ &= \int_0^\infty \int_0^t [(u^2 y^2) \wedge 1] dy \mu(dt) = \int_0^\infty \frac{1}{u} \int_0^{tu} [r^2 \wedge 1] dr \mu(dt). \end{aligned}$$

Similarly,

$$\frac{1}{2} \psi(u) = \int_0^\infty [1 - \cos(uy)] \nu_1(y) dy = \int_0^\infty [1 - \cos(uy)] \int_{(y, \infty)} \mu(dt) dy = \int_0^\infty \frac{1}{u} \kappa(tu) \mu(dt).$$

By these identities and (9) we obtain (8). \square

We define the maximal characteristic function $\psi^*(u) := \sup_{s \leq u} \psi(s)$, where $u \geq 0$. The following result is a version of [14, Proposition 1].

Proposition 2.

$$\psi(u) \leq \psi^*(u) \leq \pi^2 \psi(u) \quad \text{for } u \geq 0. \quad (10)$$

Proof. Since h_1 is nonincreasing, by Lemma 1 for $u \geq 0$ we have

$$\psi(u) \leq \psi^*(u) \leq 2 \sup_{0 < s \leq u} h_1\left(\frac{1}{s}\right) = 2h_1\left(\frac{1}{u}\right) \leq \pi^2 \psi(u).$$

\square

We write $f(x) \approx g(x)$ and say f and g are comparable if $f, g \geq 0$ and there is a positive number C , called *comparability constant*, such that $C^{-1}f(x) \leq g(x) \leq Cf(x)$ for all x . We write $C = C(a, \dots, z)$ to indicate that C may be so chosen to depend only on a, \dots, z . We say the comparison is absolute if the constant is absolute. Noteworthy, while ψ is comparable to a nondecreasing function, it need not be nondecreasing itself. For instance, if $\psi(u) = u + 3\pi[1 - (\sin u)/u]$, then $\psi'(2\pi) = -\frac{1}{2} < 0$.

The following conclusion may be interpreted as relation of "scale" and "frequency".

Corollary 3. *We have $h(r) \approx h_1(r) \approx \psi(1/r) \approx \psi^*(1/r)$ for $r > 0$.*

Proof. The constant in the leftmost comparison depends only on the dimension, see (7). The other comparisons are absolute, by Lemma 1 and Proposition 2. \square

By Corollary 3 and definitions of L_1 , L and h , we obtain the following inequality,

$$L_1(r) \leq L(r) < h(r) \leq C\psi^*(1/r), \quad r > 0. \quad (11)$$

Our main goal is to describe asymptotics of $\nu(x)$ and $p_t(x)$ in terms of ψ^* . We start with analysis of the Laplace transform of the integral tails of p_t . For reasons which shall become clear in the proof of the next result, we choose the following parametrization of the tails,

$$f_t(\rho) = \mathbb{P}(|X_t| \geq \sqrt{\rho}) = \mathbb{P}(|X_t|^2 > \rho), \quad \rho \geq 0, \quad t > 0. \quad (12)$$

We consider the Laplace transform of f_t ,

$$\mathcal{L}f_t(\lambda) = \int_0^\infty e^{-\lambda\rho} f_t(\rho) d\rho, \quad \lambda \geq 0.$$

Lemma 4. *There is a constant $C_1 = C_1(d)$ such that*

$$C_1^{-1} \frac{1}{\lambda} \left(1 - e^{-t\psi^*(\sqrt{\lambda})}\right) \leq \mathcal{L}f_t(\lambda) \leq C_1 \frac{1}{\lambda} \left(1 - e^{-t\psi^*(\sqrt{\lambda})}\right), \quad \lambda > 0.$$

Proof. By Fubini's theorem, $\int_{\mathbb{R}^d} \hat{h}(x)k(x)dx = \int_{\mathbb{R}^d} h(x)\hat{k}(x)dx$ for integrable functions h, k . By this, (12) and change of variables we obtain,

$$\begin{aligned} \lambda \mathcal{L}f_t(\lambda) &= \mathbb{E}(1 - e^{-\lambda|X_t|^2}) \\ &= 1 - \int_{\mathbb{R}^d} e^{-\lambda|x|^2} p_t(x) dx = (4\pi)^{-d/2} \int_{\mathbb{R}^d} \left(1 - e^{-t\psi(x\sqrt{\lambda})}\right) e^{-|x|^2/4} dx. \end{aligned}$$

By [15, Theorem 2.7],

$$\psi(su) \leq \psi^*(su) \leq 2(s^2 + 1)\psi^*(u) \quad s, u \geq 0. \quad (13)$$

(The estimate may usually be improved for specific ψ .) We also note that

$$1 - e^{-bt} \leq b(1 - e^{-t}), \quad t \geq 0, \quad b \geq 1, \quad (14)$$

and we obtain,

$$\begin{aligned} \lambda \mathcal{L}f_t(\lambda) &\leq (4\pi)^{-d/2} \int_{\mathbb{R}^d} \left(1 - e^{-2t(|x|^2+1)\psi^*(\sqrt{\lambda})}\right) e^{-|x|^2/4} dx \\ &\leq 2(2d+1) \left(1 - e^{-t\psi^*(\sqrt{\lambda})}\right). \end{aligned} \quad (15)$$

On the other hand, if $|x| \geq 1$, then $\psi(x\sqrt{\lambda}) \geq \psi^*(|x|\sqrt{\lambda})/\pi^2 \geq \psi^*(\sqrt{\lambda})/\pi^2$ by (10). Thus,

$$\lambda \mathcal{L}f_t(\lambda) \geq (4\pi)^{-d/2} \int_{B_1^c} e^{-|x|^2/4} dx \left(1 - e^{-t\psi^*(\sqrt{\lambda})/\pi^2}\right) \geq \frac{\Gamma(d/2, 1/4)}{\pi^2 \Gamma(d/2)} \left(1 - e^{-t\psi^*(\sqrt{\lambda})}\right),$$

where we use (14) and the upper incomplete gamma function $\Gamma(\cdot, \cdot)$. \square

The upper bounds for tails shall follow from this auxiliary lemma.

Lemma 5. *If f is nonnegative and nonincreasing, then for $n, m = 0, 1, 2, \dots$ and $r > 0$,*

$$f(r) \leq \frac{1}{\gamma(n+m+1, 1)} r^{-n-m-1} |(\mathcal{L}[s^m f])^{(n)}(r^{-1})|.$$

Proof. If $u > 0$ and $r = u^{-1}$, then

$$\begin{aligned} u^{n+m+1} |(\mathcal{L}[s^m f])^{(n)}(u)| &= u^{n+m+1} |(-1)^n \mathcal{L}[s^{n+m} f](u)| \\ &\geq u^{n+m+1} \int_0^{u^{-1}} e^{-su} s^{n+m} f(s) ds \geq f(u^{-1}) \int_0^{u^{-1}} u(us)^{n+m} e^{-su} ds \\ &= f(u^{-1}) \int_0^1 u^{n+m} e^{-u} du = f(u^{-1}) \gamma(n+m+1, 1), \end{aligned}$$

where we use the lower incomplete gamma function $\gamma(\cdot, \cdot)$. \square

The following estimate results from (15) and Lemma 5 with $n = m = 0$.

Corollary 6. *For $r > 0$ we have $\mathbb{P}(|X_t| \geq r) \leq \frac{2e}{e-1} (2d+1) (1 - e^{-t\psi^*(1/r)})$.*

Here is a general upper bound for density of unimodal Lévy process. (As we shall see in Theorem 21 and 26, a reverse inequality often holds, too.)

Corollary 7. *There is $C = C(d)$ such that $p_t(x) \leq Ct\psi^*(1/|x|)/|x|^d$ for $x \in \mathbb{R}^d \setminus \{0\}$.*

Proof. By radial monotonicity of $y \mapsto p_t(y)$, (13) and Corollary 6,

$$p_t(x) \leq \frac{\mathbb{P}(|x|/2 \leq |X_t| < |x|)}{|B_{|x|} \setminus B_{|x|/2}|} \leq \frac{d}{(1-2^{-d})\omega_d} \mathbb{P}\left(|X_t| \geq \frac{|x|}{2}\right) |x|^{-d} \leq C \frac{t\psi^*(1/|x|)}{|x|^d}.$$

\square

Since $p_t(x)/t \rightarrow \nu(x)$ vaguely on $\mathbb{R}^d \setminus \{0\}$, we also obtain

$$\nu(x) \leq C \frac{\psi^*(1/|x|)}{|x|^d}, \quad x \in \mathbb{R}^d \setminus \{0\}. \quad (16)$$

Tracking constants, e.g., for the isotropic α -stable Lévy process addressed in (4), we get

$$p_t(x) \leq \frac{d4^\alpha \Gamma\left(\frac{d+\alpha}{2}\right)}{2(1-2^{-d})\pi^{d/2}\gamma(1, 1)} \frac{t}{|x|^{d+\alpha}}, \quad t > 0, x \in \mathbb{R}^d. \quad (17)$$

(In fact, for this constant we override (13) by $\psi(su) = s^\alpha \psi(u)$ in the proof of Lemma 4.)

3 Weak scaling and monotonicity

For reader's convenience we shall give a short survey of almost monotone and weakly scaling functions. The results may be considered folklore [5, 18], but the actual variants which we need may be difficult to find.

Let $\phi : I \rightarrow [0, \infty]$, for a connected set $I \subset [-\infty, \infty]$. First, we call ϕ *almost increasing* if there is (oscillation factor) $c \in (0, 1]$ such that $c\phi(x) \leq \phi(y)$ for $x, y \in I$, $x \leq y$. Let

$$\phi^*(y) = \sup\{\phi(x) : x \in I, x \leq y\}, \quad y \in I.$$

We easily check that ϕ^* is nondecreasing, $\phi \leq \phi^*$ and the following result holds.

Lemma 8. *ϕ is almost increasing with oscillation factor c if and only if $c\phi^* \leq \phi$.*

E.g. the Lévy-Khintchine exponent ψ in Proposition 2 is almost increasing with factor $1/\pi^2$. On the other hand, if there is $C \in [1, \infty)$ such that $C\phi(x) \geq \phi(y)$ for $x, y \in I$, $x \leq y$, then we call ϕ *almost decreasing* (with oscillation factor C). Let

$$\phi_*(x) = \sup\{\phi(y) : y \in I, y \geq x\}, \quad x \in I.$$

We easily check that ϕ_* is nonincreasing, $\phi \leq \phi_*$ and the following result holds.

Lemma 9. *ϕ is almost decreasing with oscillation factor C if and only if $\phi_* \leq C\phi$.*

We note that ϕ is almost increasing on I with factor c if and only if $1/\psi$ is almost decreasing on I with factor $1/c$. Here is another simple observation which we give without proof.

Lemma 10. *Assume that sets $I_1, I_2, I = I_1 \cup I_2 \subset \mathbb{R}$ are connected. If ϕ is almost increasing (decreasing) on I_1 with factor c' (C'), almost increasing (decreasing) on I_2 with factor c'' (C''), then ϕ is almost increasing (decreasing) on I with factor $c = c'c''$ ($C = C'C''$).*

We now specialize to $I = (\underline{\theta}, \infty)$, where $\underline{\theta} \in [0, \infty)$. We say that ϕ satisfies the weak lower scaling condition (at infinity) if there are numbers $\underline{\alpha} \in \mathbb{R}$ and $\underline{c} \in (0, 1]$, such that

$$\phi(\lambda\theta) \geq \underline{c}\lambda^{\underline{\alpha}}\phi(\theta) \quad \text{for } \lambda \geq 1, \quad \theta \in I. \quad (18)$$

In short we say that ϕ satisfies $\text{WLSC}(\underline{\alpha}, \underline{\theta}, \underline{c})$ and write $\phi \in \text{WLSC}(\underline{\alpha}, \underline{\theta}, \underline{c})$. If $\phi \in \text{WLSC}(\underline{\alpha}, 0, \underline{c})$, then we say that ϕ satisfies the *global* weak lower scaling condition.

Similarly, we consider $I = (\bar{\theta}, \infty)$, where $\bar{\theta} \in [0, \infty)$. The weak upper scaling condition holds if there are numbers $\bar{\alpha} \in \mathbb{R}$ and $\bar{C} \in [1, \infty)$ such that

$$\phi(\lambda\theta) \leq \bar{C}\lambda^{\bar{\alpha}}\phi(\theta) \quad \text{for } \lambda \geq 1, \quad \theta \in I. \quad (19)$$

In short, $\phi \in \text{WUSC}(\bar{\alpha}, \bar{\theta}, \bar{C})$. For *global* weak upper scaling we require $\bar{\theta} = 0$ in (19).

The following clarification is an analogue of [5, Theorem 2.2.2].

Lemma 11. *We have $\phi \in \text{WLSC}(\underline{\alpha}, \underline{\theta}, \underline{c})$ if and only if $\phi(\theta) = \kappa(\theta)\theta^{\underline{\alpha}}$ and κ is almost increasing on $(\underline{\theta}, \infty)$ with oscillation factor \underline{c} . Similarly, $\phi \in \text{WUSC}(\bar{\alpha}, \bar{\theta}, \bar{C})$ if and only if $\phi(\theta) = \kappa(\theta)\theta^{\bar{\alpha}}$ and κ is almost decreasing on $(\bar{\theta}, \infty)$ with oscillation factor \bar{C} .*

Proof. Let $\underline{\theta} \in [0, \infty)$, $I = (\underline{\theta}, \infty)$ and $\phi \in \text{WLSC}(\underline{\alpha}, \underline{\theta}, \underline{c})$. Let $\kappa(\theta) = \phi(\theta)\theta^{-\alpha}$ on I . If $\underline{\theta} < \eta \leq \theta$ and $\lambda = \theta/\eta$, then

$$\kappa(\theta) = \phi(\lambda\eta)(\lambda\eta)^{-\alpha} \geq \underline{c}\lambda^{\alpha}\phi(\eta)(\lambda\eta)^{-\alpha} = \underline{c}\kappa(\eta).$$

Thus, κ is almost increasing. On the other hand, if $\phi(\theta) = \kappa(\theta)\theta^{\alpha}$ and κ is almost increasing with factor \underline{c} , then for $\lambda \geq 1$ and $\theta \in I$ we have

$$\phi(\lambda\theta) = \kappa(\lambda\theta)(\lambda\theta)^{\alpha} \geq \underline{c}\kappa(\theta)(\lambda\theta)^{\alpha} = \underline{c}\lambda^{\alpha}\phi(\theta).$$

Thus, $\phi \in \text{WLSC}(\underline{\alpha}, \underline{\theta}, \underline{c})$. The proof of the second part of the statement is left to the reader. \square

In particular by Lemma 11, $\text{WLSC}(0, \underline{\theta}, \underline{c})$ characterizes almost increasing functions on $(\underline{\theta}, \infty)$, and $\text{WUSC}(0, \bar{\theta}, \bar{C})$ characterizes almost decreasing functions on $(\bar{\theta}, \infty)$.

For example, $h, h_1 \in \text{WUSC}(0, 0, 1) \cap \text{WLSC}(-2, 0, 1)$, see Remark 1.

Remark 2. If $\phi \in \text{WLSC}(\underline{\alpha}, \underline{\theta}, \underline{c})$ and $c\phi \leq \varphi \leq C\phi$, then $\varphi \in \text{WLSC}(\underline{\alpha}, \underline{\theta}, \underline{c}C/c)$. Similarly, if $\phi \in \text{WUSC}(\bar{\alpha}, \bar{\theta}, \bar{C})$ and $c\phi \leq \varphi \leq C\phi$, then $\varphi \in \text{WUSC}(\bar{\alpha}, \bar{\theta}, \bar{C}C/c)$.

As $\underline{\theta}$ or $\bar{\theta}$ decrease, the scaling conditions tighten. Here is a loosening observation.

Lemma 12. Let $\phi : [\theta_1, \infty) \rightarrow (0, \infty)$ be nondecreasing and nonzero. If $0 < \theta_1 < \underline{\theta}$ and $\phi \in \text{WLSC}(\underline{\alpha}, \underline{\theta}, \underline{c})$, then $\phi \in \text{WLSC}(\underline{\alpha}, \theta_1, c_1)$ with $c_1 = \underline{c}(\theta_1/\underline{\theta})^{|\underline{\alpha}|} \phi(\theta_1)/\phi(\underline{\theta})$. If rather $0 < \theta_1 < \bar{\theta}$ and $\phi \in \text{WUSC}(\bar{\alpha}, \bar{\theta}, \bar{C})$, then $\phi \in \text{WUSC}(\bar{\alpha}, \theta_1, C_1)$ with $C_1 = \bar{C}(\bar{\theta}/\theta_1)^{|\bar{\alpha}|} \phi(\bar{\theta})/\phi(\theta_1)$.

Proof. In view of Lemma 10 and Lemma 11, it is enough to study $\kappa(\theta) = \phi(\theta)\theta^{-\alpha}$ on $[\theta_1, \underline{\theta}]$. If $\theta_1 \leq \theta \leq \underline{\theta}$, then $\phi(\theta_1)(\underline{\theta}^{-\alpha} \wedge \theta_1^{-\alpha}) \leq \kappa(\theta) \leq \phi(\underline{\theta})(\theta_1^{-\alpha} \vee \underline{\theta}^{-\alpha})$. This gives the first implication and the second obtains similarly. \square

Remark 3. Let $\phi \geq 0$ be continuous an increase to infinity. If $\phi \in \text{WLSC}(\underline{\alpha}, \underline{\theta}, \underline{c})$ [$\text{WUSC}(\bar{\alpha}, \bar{\theta}, \bar{C})$], then $\phi^{-1} \in \text{WUSC}(1/\underline{\alpha}, \phi(\underline{\theta}), \underline{c}^{-1/\underline{\alpha}})$ [$\text{WLSC}(1/\bar{\alpha}, \phi(\bar{\theta}), \bar{C}^{-1/\bar{\alpha}})$, resp.]. Indeed, since ϕ is increasing, by the scaling for $\lambda \geq 1$ and $\theta > \underline{\theta}$ we have

$$\lambda\phi^{-1}(\phi(\theta)) = \lambda\theta = \phi^{-1}(\phi(\lambda\theta)) \geq \phi^{-1}(\underline{c}\lambda^{\alpha}\phi(\theta)).$$

Thus for (arbitrary) $y = \phi(\theta) > \phi(\underline{\theta})$, if $s = \underline{c}\lambda^{\alpha} \geq \underline{c}$, in particular if $s \geq 1$, then

$$\underline{c}^{-1/\underline{\alpha}}s^{1/\underline{\alpha}}\phi^{-1}(y) \geq \phi^{-1}(sy).$$

Similarly, if $\phi \in \text{WUSC}(\bar{\alpha}, \bar{\theta}, \bar{C})$, then

$$\bar{C}^{-1/\bar{\alpha}}s^{1/\bar{\alpha}}\phi^{-1}(y) \leq \phi^{-1}(sy), \quad y > \phi(\bar{\theta}), \quad s \geq \bar{C}.$$

For $1 \leq s < \bar{C}$, by monotonicity of ϕ^{-1} ,

$$\bar{C}^{-1/\bar{\alpha}}s^{1/\bar{\alpha}}\phi^{-1}(y) \leq \phi^{-1}(y) \leq \phi^{-1}(sy), \quad y > \phi(\bar{\theta}).$$

This proves our claim.

Remark 4. We also note that $\phi \in \text{WLSC}(\underline{\alpha}, \underline{\theta}, \underline{c})$ if and only if $1/\phi \in \text{WUSC}(-\underline{\alpha}, \underline{\theta}, 1/\underline{c})$. Similarly, $\phi(t) \in \text{WLSC}(\underline{\alpha}, 0, \underline{c})$ if and only if $\phi(1/t) \in \text{WUSC}(-\underline{\alpha}, 0, 1/\underline{c})$. For instance, our Lévy-Khintchine exponent ψ has global upper scaling: by Remark 1, Lemma 11 and Lemma 1, $\psi, \psi^* \in \text{WUSC}(2, 0, \pi^2)$, and $\psi \in \text{WLSC}(0, 0, 1/\pi^2)$. In fact, by (13) we have

$$\psi^*(u) \leq \psi^*(\lambda u) \leq 4\lambda^2\psi^*(u), \quad \lambda \geq 1, u \geq 0,$$

and so $\psi^* \in \text{WUSC}(2, 0, 4)$. Of course, $\psi^* \in \text{WLSC}(0, 0, 1)$, meaning that ψ^* is nondecreasing.

4 Scaling of the Lévy-Khintchine exponent

We shall study consequences of scaling of the Lévy-Khintchine exponent ψ of (isotropic) unimodal pure-jump Lévy process X with nonzero Lévy measure ν . For economy of notation, in the sequel we only consider (assume) scaling exponents $\underline{\alpha}, \bar{\alpha}$ satisfying:

$$0 < \underline{\alpha} < 2 \quad \text{and} \quad 0 < \bar{\alpha} < 2. \quad (20)$$

4.1 Examples

The Lévy-Khintchine (characteristic) exponents of unimodal convolution semigroups which we present in this section all have lower or upper scaling suggested by (20). This can be verified in each case by using Lemma 11. While discussing the exponents, we shall also make connection to subordinators, special Bernstein functions and complete Bernstein functions, because they are intensely used in recent study of subordinate Brownian motions, a wide and diverse family of unimodal Lévy processes cf. [18]. The reader may find definitions and comprehensive information on these functions in [26]. When discussing subordinators we usually let $\varphi(\lambda)$ denote their Laplace exponent, and then $\psi(x) = \varphi(|x|^2)$ is the Lévy-Khintchine exponent of the corresponding subordinate Brownian motion. We focus on scaling properties of ψ .

1. Let $\psi(u) = \sum_{i=1}^n c_i u^{\alpha_i}$, where $u \geq 0$, $n \in \mathbb{N}$, $c_i > 0$, $\alpha_i \in (0, 2)$. This ψ is the Lévy-Khintchine exponent of $X = \sum_{i=1}^n c_i^{1/\alpha_i} X_{(i)}$, where $X_{(i)}$ are independent isotropic α_i -stable Lévy processes (in \mathbb{R}^d). Let $\underline{\alpha} = \min \alpha_i$, $\bar{\alpha} = \max \alpha_i$. Then for every $\varepsilon > 0$, $\psi \in \text{WLSC}(\bar{\alpha}, \varepsilon, \underline{c})$, where $\underline{c} = c_j / (\sum_{i=1}^n c_i \varepsilon^{\alpha_i - \bar{\alpha}})$ and j is such that $\bar{\alpha} = \alpha_j$. Even simpler, $\psi \in \text{WUSC}(\bar{\alpha}, 0, 1)$ and $\psi \in \text{WLSC}(\underline{\alpha}, 0, 1)$. The latter is also true if $\psi(u) = \int u^\alpha \mu(d\alpha)$, where μ is a finite measure on $[\underline{\alpha}, \bar{\alpha}]$ (and $0 < \underline{\alpha} < \bar{\alpha} < 2$). More generally, let A be a (Borel) set and let $\psi(s, u) = \psi_s(u)$ satisfy $\text{WLSC}(\underline{\alpha}, \underline{\theta}, \underline{c})$ [$\text{WUSC}(\bar{\alpha}, \bar{\theta}, \bar{C})$] for $s \in A$, and

$$\psi_s(\xi) = \psi_s(|\xi|) = \int_{\mathbb{R}^d} (1 - \cos \langle \xi, x \rangle) \nu_s(dx), \quad \xi \in \mathbb{R}^d,$$

for some unimodal Lévy measures ν_s (to wit, $\nu_s(\cdot)$ is assumed jointly Borel). Let μ be a measure on A such that $\int_A \psi_s(1) \mu(ds) < \infty$. Then $\psi(x) = \int_A \psi_s(x) \mu(ds)$, which is finite by Remark 2, is the Lévy-Khintchine exponent of a unimodal Lévy process and we have $\psi \in \text{WLSC}(\underline{\alpha}, \underline{\theta}, \underline{c})$ [$\text{WUSC}(\bar{\alpha}, \bar{\theta}, \bar{C})$, respectively].

2. Let $\varphi(\lambda) = \int_0^\infty (1 - e^{-\lambda u}) \mu(dr)$ be a Bernstein function [26], i.e. the Laplace exponent of a subordinator η [26, 4, 25, 1], and let Y be an independent (isotropic) unimodal Lévy process with characteristic exponent χ . Then the process $X_t = Y_{\eta_t}$ is unimodal and has the characteristic exponent $\psi(x) = \varphi(\chi(x))$ [26]. If $\chi \in \text{WUSC}(\bar{\alpha}_1, \bar{\theta}, \bar{C}_1)$ and $\varphi \in \text{WUSC}(\bar{\alpha}_2, \chi(\bar{\theta}), \bar{C}_2)$, then $\psi \in \text{WUSC}(\bar{\alpha}_1 \bar{\alpha}_2, \bar{\theta}, \bar{C}_1^{\bar{\alpha}_2} \bar{C}_2)$. From concavity of Bernstein functions it also follows that if $\chi \in \text{WLSC}(\underline{\alpha}_1, \underline{\theta}, \underline{c}_1)$, $\theta_* = \inf_{\theta \geq \underline{\theta}} \chi(\theta)$ and $\varphi \in \text{WLSC}(\underline{\alpha}_2, \theta_*, \underline{c}_2)$, then $\psi \in \text{WLSC}(\underline{\alpha}_1 \underline{\alpha}_2, \underline{\theta}, \underline{c}_1 \underline{c}_2)$. We always have $\theta_* \geq \chi(\underline{\theta})/\pi^2$, see Proposition 2, and often $\theta_* = \chi(\underline{\theta})$. Of particular interest here is $\chi(\xi) = |\xi|^2$, i.e. $Y_t = B_{2t}$, where B is the Brownian motion in \mathbb{R}^d . The process X is then called a subordinate Brownian motion. Furthermore, it is called *special* subordinate Brownian motion if the subordinator is *special* (i.e. given by a special Bernstein function), and it is called *complete* subordinate Brownian motion if the subordinator is *complete* [26]. The (unimodal) Lévy measure density of Y is given by the formula

$$\nu(x) = \int_0^\infty (4\pi t)^{-d/2} e^{-\frac{|x|^2}{4t}} \mu(dt), \quad (21)$$

and its Lévy-Khintchine exponent $\varphi(|\xi|^2)$ is in $\text{WUSC}(2\bar{\alpha}_1, \bar{\theta}^2, \bar{C}_1^2)$ or $\text{WLSC}(2\underline{\alpha}_1, \underline{\theta}^2, \underline{C}_1^2)$, respectively.

3. Let $\psi(\xi) = |\xi|^\alpha \log^\beta(1 + |\xi|^\gamma)$, where $\gamma, \alpha, \alpha + 2\beta \in (0, 2)$. If $\varepsilon > 0$, then $\psi \in \text{WUSC}(\alpha + \varepsilon, 1, \bar{C}) \cap \text{WLSC}(\alpha - \varepsilon, 1, \underline{C})$ for some $0 < \underline{C} \leq 1 \leq \bar{C} < \infty$. Furthermore, $\psi \in \text{WUSC}(\alpha + \gamma\beta_+, 0, 1) \cap \text{WLSC}(\alpha - \gamma\beta_-, 0, 1)$. We note that ψ is the Lévy-Khintchine exponent of a subordinate Brownian motion, see Theorem 12.14, Proposition 7.10, Proposition 7.1, Corollary 7.9, Section 13 and examples 1 and 26 from Section 15.2 in [26]. Many more examples related to subordinate Brownian motions readily follow from [26, Section 15].
4. Let $\mu(r) = (1 - \sin r)r^{-1-\alpha/2}$, where $\alpha \in (0, 2)$ and $\psi(\xi) = \int_0^\infty (1 - e^{-r|\xi|^2})\mu(r)dr$; in fact, let $f(t)$ be a nonnegative bounded function on $(0, \infty)$ and define $\mu(r) = f(r)r^{-1-\alpha/2}$. Then $\psi(\xi) = \int_0^\infty (1 - e^{-r|\xi|^2})\mu(r)dr$ is the characteristic exponent of a subordinate Brownian motion and $\psi(\xi) \leq C|\xi|^\alpha$. We also note that

$$\psi(\xi) = \int_0^\infty (1 - e^{-r|\xi|^2})\mu(r)dr \geq e^{-1}|\xi|^2 \int_0^{|\xi|^{-2}} r\mu(r)dr + (1 - e^{-1}) \int_{|\xi|^{-2}}^\infty \mu(r)dr.$$

If $\liminf_{|\xi| \rightarrow \infty} |\xi|^{2-\alpha} \int_0^{|\xi|^{-2}} f(r)r^{-\alpha/2}dr > 0$, then $\psi(\xi) \geq c|\xi|^\alpha$, hence $\psi(\xi) \approx |\xi|^\alpha$ for $|\xi| \geq 1$. For example $f(r) = (1 - \sin r)$ and $f(r) = \sin^2(r + r^{-1})$ produce such ψ . For later use we note that these ψ may be considered Lévy-Khintchine exponents of subordinate Brownian motion but the Laplace exponents of the respective subordinators are not complete Bernstein functions, as their Lévy measure densities $\mu(r)$ are not monotone, cf. [26, Definition 6.1].

5. Let X be pure-jump unimodal with infinite Lévy measure and Lévy-Khintchine exponent ψ . Let a scaling condition with exponent $\underline{\alpha}$ or $\bar{\alpha}$ hold for ψ . For fixed $r > 0$, we let X^r be the (truncated) unimodal Lévy process obtained by multiplying the Lévy measure of X by the indicator function of the ball B_r , and let ψ_r be its Lévy-Khintchine exponent. Since $0 \leq \psi - \psi_r$ is bounded, ψ_r is comparable with ψ at infinity, and so ψ_r has (local) scaling with the same exponent as ψ . For later discussion we observe that ψ_r is not an exponent of a subordinate Brownian motion because the support of its Lévy measure is bounded [26, Proposition 10.16].
6. We may combine the two preceding examples to obtain subordinate Brownian motions which have local scaling but are not *special*, since the Lévy measure of their subordinators is being truncated. In fact, let $\alpha \in (0, 2)$ and $\mu(dr) = \sum_{k=2}^\infty \delta_{1/k}(dr) (k^{\alpha/2} - (k-1)^{\alpha/2})$, $\varphi(\lambda) = \int_0^\infty (1 - e^{-r\lambda})\mu(dr)$. This φ is not a complete or even special Bernstein function [26, Proposition 10.16]. As usual, $\psi(\xi) = \varphi(|\xi|^2)$ is the characteristic exponent of a subordinate Brownian motion. Since $\varphi(\lambda) \approx \lambda^{\alpha/2} \wedge \lambda$, we have $\psi \in \text{WUSC}(\alpha, 1, \bar{C}) \cap \text{WLSC}(\alpha, 1, \underline{C})$ for some $0 < \underline{C} \leq 1 \leq \bar{C}$.
7. Let $0 < \alpha_1 < \alpha_2 < 2$ and $u(r) = r^{\alpha_1/2-1} \vee r^{\alpha_2/2-1}$. Let $\eta(\lambda) = \mathcal{L}u(\lambda) = \lambda^{-\alpha_1/2}\gamma(\alpha_1/2, \lambda) + \lambda^{-\alpha_2/2}\Gamma(\alpha_2/2, \lambda)$ and $\varphi(\lambda) = 1/\eta(\lambda)$. Note that $\varphi(\lambda) \approx \lambda^{\alpha_1/2} \wedge \lambda^{\alpha_2/2}$. Therefore $\varphi(|x|^2) \in \text{WUSC}(\alpha_2, 0, \bar{C})$ and $\text{WLSC}(\alpha_1, 0, \underline{C})$ for some $0 < \underline{C} \leq 1 \leq \bar{C}$. It is shown in [26, Example 10.18(i)] that φ is a special Bernstein function but not a complete Bernstein function. Moreover, the Lévy measure of φ is not known and so previous methods of estimating transition densities of the resulting subordinate Brownian motion do not yet apply [18].

4.2 Estimates

The following estimate is a version of [33, Theorem 7 (ii) (b)] with explicit constants.

Lemma 13. *Let $f \geq 0$ be nonincreasing, $\beta > 0$ and $\mathcal{L}f \in WUSC(-\beta, \bar{\theta}, \bar{C})$. There is $b = b(\beta, \bar{C}) \in (0, 1)$ such that*

$$f(r) \geq \frac{b}{2} e^b r^{-1} \mathcal{L}f(r^{-1}), \quad 0 < r < b/\bar{\theta}.$$

Proof. Let $0 < b < 1$. If $u > \bar{\theta}$, then by Lemma 5 and the upper scaling (with $\lambda = s^{-1}/u$),

$$\begin{aligned} u\mathcal{L}f(u) &= u \int_0^{bu^{-1}} e^{-us} f(s) ds + u \int_{bu^{-1}}^\infty e^{-us} f(s) ds \\ &\leq \frac{u}{\gamma(1, 1)} \int_0^{bu^{-1}} e^{-us} \mathcal{L}f(s^{-1}) s^{-1} ds + f(bu^{-1}) \int_{bu^{-1}}^\infty e^{-us} u ds \\ &\leq \frac{u}{\gamma(1, 1)} \int_0^{bu^{-1}} \bar{C} (us)^\beta \mathcal{L}f(u) e^{-us} s^{-1} ds + f(bu^{-1}) e^{-b} \\ &= \bar{C} \frac{\gamma(\beta, b)}{\gamma(1, 1)} u \mathcal{L}f(u) + f(bu^{-1}) e^{-b}. \end{aligned}$$

If $2\bar{C}\gamma(\beta, b) \leq \gamma(1, 1) = 1 - e^{-1}$, then $f(bu^{-1}) \geq e^b u \mathcal{L}f(u)/2$. We change variables: $r = bu^{-1}$. Since $\mathcal{L}f$ is decreasing,

$$f(r) \geq \frac{b}{2} e^b r^{-1} \mathcal{L}f(br^{-1}) \geq \frac{b}{2} e^b r^{-1} \mathcal{L}f(r^{-1}), \quad r < b/\bar{\theta}.$$

□

Lemma 14. *$C = C(d)$ exists such that if $\psi \in WUSC(\bar{\alpha}, \bar{\theta}, \bar{C})$ and $a = [(2 - \bar{\alpha})C]^{\frac{2}{2-\bar{\alpha}}} \bar{C}^{\frac{\bar{\alpha}-2}{2}}$, then*

$$\mathbb{P}(|X_t| \geq r) \geq a \left(1 - e^{-t\psi^*(1/r)}\right), \quad 0 < r < \sqrt{a}/\bar{\theta}.$$

Proof. By Lemma 4, Proposition 2 and (14), for $\lambda \geq 1$ and $u \geq \bar{\theta}^2$ we have,

$$\frac{\mathcal{L}f_t(\lambda u)}{\mathcal{L}f_t(u)} \leq C_1^2 \lambda^{-1} \frac{1 - e^{-\pi^2 t \psi(\sqrt{\lambda u})}}{1 - e^{-t\psi(\sqrt{u})}} \leq C_1^2 \lambda^{-1} \frac{1 - e^{-\pi^2 t \psi(\sqrt{u})} \bar{C} \lambda^{\bar{\alpha}/2}}{1 - e^{-t\psi(\sqrt{u})}} \leq \pi^2 C_1^2 \bar{C} \lambda^{\bar{\alpha}/2-1}.$$

Thus, $\mathcal{L}f_t \in WUSC(\bar{\alpha}/2 - 1, \bar{\theta}^2, \pi^2 C_1^2 \bar{C})$. By Lemma 13 and Lemma 4,

$$\mathbb{P}(|X_t| \geq r) = f_t(r^2) \geq \frac{b}{2} e^b r^{-2} \mathcal{L}f_t(r^{-2}) \geq \frac{b}{2C_1} \left(1 - e^{-t\psi^*(1/r)}\right), \quad r^2 < b/\bar{\theta}^2.$$

Here $b \in (0, 1)$ is such that $2\pi^2 C_1^2 \bar{C} \gamma(1 - \bar{\alpha}/2, b) \leq 1 - e^{-1}$, see the proof of Lemma 13. Since $\gamma(1 - \bar{\alpha}/2, b) < b^{1-\bar{\alpha}/2}/(1 - \bar{\alpha}/2)$, we may take $b = \left(\frac{1-\bar{\alpha}/2}{2\pi^2 C_1^2 \bar{C}}\right)^{1/(1-\bar{\alpha}/2)}$ and $a = b/(2C_1) < 1$. □

Since $\lim_{t \rightarrow 0^+} \mathbb{P}(|X_t| \geq r)/t = \nu(B_r^c) = L(r)$ for $r > 0$, we obtain the following result.

Corollary 15. *If ψ satisfies $WUSC(\bar{\alpha}, \bar{\theta}, \bar{C})$ and a is from Lemma 14, then*

$$L(r) \geq a\psi^*(1/r), \quad 0 < r < \sqrt{a}/\bar{\theta}.$$

We recall that a reverse inequality is valid more universally, cf. (11).

In view of (18) and (20), the lower weak scaling implies power-type asymptotic growth of ψ and so we can use Fourier inversion to estimate p_t . Here is a preparation.

Lemma 16. *Let $\alpha > 0$ and Ψ be an increasing function on $[0, \infty)$, such that $\Psi(0) = 0$ and $\Psi \in WLS C(\alpha, \underline{\theta}, \underline{c})$. There is $C_2 = C_2(d, \alpha)$ such that if $0 < K < \infty$, $t > 0$ and $t\Psi(\underline{\theta}) < K$, then*

$$\int_{\mathbb{R}^d} e^{-t\Psi(|\xi|)} d\xi \leq C_2 (1 \vee (\underline{c}K)^{-d/\alpha-1}) (\Psi^{-1}(K/t))^d.$$

Proof. Since Ψ increases and scales, it is unbounded. Hence, for $t > 0$,

$$\int_{\mathbb{R}^d} e^{-t\Psi(|\xi|)} d\xi = \sum_{n=1}^{\infty} \int_{\underline{c}K(n-1) \leq t\Psi(|\xi|) < \underline{c}Kn} e^{-t\Psi(|\xi|)} d\xi \leq \frac{\omega_d}{d} \sum_{n=1}^{\infty} [\Psi^{-1}(\underline{c}Kn/t)]^d e^{-\underline{c}K(n-1)}.$$

Also, if $\Psi(\underline{\theta}) < K/t$, then $\Psi^{-1}(K/t) > \underline{\theta}$. By lower scaling, for $n \geq 1$,

$$\Psi(n^{1/\alpha} \Psi^{-1}(K/t)) \geq \underline{c}n \Psi(\Psi^{-1}(K/t)) = \underline{c}nK/t,$$

which yields

$$n^{1/\alpha} \Psi^{-1}(K/t) \geq \Psi^{-1}(\underline{c}Kn/t).$$

We obtain,

$$\int_{\mathbb{R}^d} e^{-t\Psi(|\xi|)} d\xi \leq [\Psi^{-1}(K/t)]^d \frac{\omega_d}{d} \sum_{n=1}^{\infty} n^{d/\alpha} e^{-\underline{c}K(n-1)}.$$

For $\rho \geq 0$ and $u > 0$ we have

$$\begin{aligned} S(u, \rho) &= \sum_{n=1}^{\infty} n^\rho e^{-u(n-1)} = e^{2u} \sum_{n=1}^{\infty} n^\rho e^{-u(n+1)} \\ &\leq e^{2u} \int_0^\infty x^\rho e^{-ux} dx = e^{2u} u^{-1-\rho} \Gamma(\rho+1). \end{aligned}$$

Since $S(u, \rho)$ is decreasing in u , we also have $S(u, \rho) \leq S(1, \rho)$ for $u \geq 1$, thus

$$S(u, \rho) \leq e^2 \Gamma(\rho+1) (1 \vee u^{-1-\rho}), \quad u > 0.$$

The proof is complete: $\sum_{n=1}^{\infty} n^{d/\alpha} e^{-\underline{c}K(n-1)} \leq e^2 \Gamma(d/\alpha+1) (1 \vee (\underline{c}K)^{-d/\alpha-1})$. \square

For a continuous nondecreasing function $\phi : [0, \infty) \rightarrow [0, \infty)$, such that $\phi(0) = 0$, we let $\phi(\infty) = \lim_{s \rightarrow \infty} \phi(s)$ and we define the generalized inverse $\phi^- : [0, \infty) \rightarrow [0, \infty)$,

$$\phi^-(u) = \inf\{s \geq 0 : \phi(s) \geq u\}, \quad 0 \leq u < \infty,$$

with the convention that $\inf \emptyset = \infty$. The function is nondecreasing and càglàd where finite. Notice that $\phi(\phi^-(u)) = u$ for $u \in [0, \phi(\infty)]$ and $\phi^-(\phi(s)) \leq s$ for $s \in [0, \infty)$. Also, if $\varphi : [0, \infty) \rightarrow [0, \infty)$, $\varphi(0) = 0$, $c > 0$ and $c\phi \leq \varphi$, then $\phi^-(u) \geq \varphi^-(cu)$, $u \geq 0$. Below we shall often consider the (unbounded) characteristic exponent ψ of a unimodal Lévy process with infinite Lévy measure and its (comparable) maximal function ψ^* , and denote

$$\psi^- = (\psi^*)^-.$$

This short notation is motivated by the following equality,

$$\inf\{s \geq 0 : \psi(s) \geq u\} = \inf\{s \geq 0 : \psi^*(s) \geq u\}, \quad 0 \leq u < \infty.$$

Note that $\psi^*(\psi^-(u)) = u$, $\psi^-(\psi^*(s)) \leq s$. The reader may find it instructive to prove the following result for $t > 0$ and $x \in \mathbb{R}^d \setminus \{0\}$.

Lemma 17. $t\psi^*(1/|x|) \geq 1$ if and only if $t\psi^*(1/|x|)|x|^{-d} \geq [\psi^-(1/t)]^d$.

In what follows it may be helpful to view $t\psi^*(1/|x|) \geq 1$ and $t\psi^*(1/|x|) < 1$ as defining "large time" and "small time" for given $x \in \mathbb{R}^d \setminus \{0\}$, respectively.

We note that $\psi^-(\psi^*(s)+) \geq s$ for $s \in [0, \infty)$, where $\psi^-(u+)$ denotes the right hand side limit of ψ^- at $u \geq 0$. Furthermore, scaling of ψ translates into scaling of ψ^- as follows.

Lemma 18. If $\psi \in \text{WLSC}(\underline{\alpha}, 0, \underline{c})$, then $\psi^- \in \text{WLSC}(1/2, 0, (\underline{c}/\pi^4)^{1/\underline{\alpha}}) \cap \text{WUSC}(1/\underline{\alpha}, 0, (\pi^3/\underline{c})^{2/\underline{\alpha}})$.

Proof. We let $\Psi(s) = h_1(s^{-1})$, $s > 0$. Lemma 11 and Remark 1 yield $\Psi \in \text{WUSC}(2, 0, 1)$. By Lemma 1 and Remark 2, $\Psi \in \text{WLSC}(\underline{\alpha}, 0, \underline{c}/\pi^2)$. Remark 3 implies

$$\Psi^{-1} \in \text{WLSC}(1/2, 0, 1) \cap \text{WUSC}(1/\underline{\alpha}, 0, (\pi^2/\underline{c})^{1/\underline{\alpha}}).$$

By Lemma 1 and the above scaling,

$$\Psi^{-1}(s/2) \leq \psi^-(s) \leq \Psi^{-1}(s\pi^2/2) \leq (\pi^4/\underline{c})^{1/\underline{\alpha}} \Psi^{-1}(s/2).$$

Hence, by Remark 2, $\psi^- \in \text{WLSC}(1/2, 0, (\underline{c}/\pi^4)^{1/\underline{\alpha}}) \cap \text{WUSC}(1/\underline{\alpha}, 0, (\pi^3/\underline{c})^{2/\underline{\alpha}})$. \square

We shall use Fourier inversion and (1) to estimate $p_t(0)$: if, say, $\lim_{|\xi| \rightarrow \infty} \psi(\xi)/\ln(|\xi|) = \infty$, then $e^{-t\psi(\xi)}$ is integrable for $t > 0$, $\mathbb{R}^d \ni x \mapsto p_t(x)$ may be assumed continuous, and we have,

$$p_t(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-t\psi(\xi)} e^{-i\langle \xi, x \rangle} d\xi.$$

In particular,

$$p_t(0) \geq (2\pi)^{-d} \int_{B(0, \psi^-(1/t))} e^{-t\psi^*(|\xi|)} d\xi \geq (2\pi)^{-d} \frac{\omega_d}{ed} [\psi^-(1/t)]^d, \quad t > 0. \quad (22)$$

Lower scaling yields a reverse inequality.

Proposition 19. If $\alpha > 0$ and $\psi \in \text{WLSC}(\alpha, \underline{\theta}, \underline{c})$, then $C = C(d, \alpha)$ exists such that

$$p_t(x) \leq C \underline{c}^{-d/\alpha-1} [\psi^-(1/t)]^d \quad \text{if } t > 0 \text{ and } t\psi^*(\underline{\theta}) < 1/\pi^2. \quad (23)$$

Proof. We let

$$\Psi(s) = h_1(s^{-1}), \quad s > 0.$$

Note that h_1 and Ψ are strictly monotone. According to Lemma 1,

$$2\Psi(s)/\pi^2 \leq \psi(s) \leq \psi^*(s) \leq 2\Psi(s),$$

hence $\Psi \in \text{WLSC}(\alpha, \underline{\theta}, \underline{c}/\pi^2)$. Furthermore, $\psi^-(2u/\pi^2) \leq \Psi^{-1}(u) \leq \psi^-(2u)$ for $u \geq 0$. Let $t > 0$. If $t\psi^*(\underline{\theta}) < 1/\pi^2$, then $(2t/\pi^2)\Psi(\underline{\theta}) < 1/\pi^2$. We apply Lemma 16 to $K = 1/\pi^2$ and (time) $2t/\pi^2$:

$$p_t(x) \leq (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-2t\Psi(|\xi|)/\pi^2} d\xi \leq (2\pi)^{-d} C_2 (\underline{c}/\pi^2)^{-d/\alpha-1} [\Psi^{-1}((2t)^{-1})]^d.$$

But $\Psi^{-1}((2t)^{-1}) \leq \psi^-(1/t)$, and the proof is complete. \square

Under the assumptions of Proposition 19, by (22) and (23) we obtain

$$e^{-1} \leq p_t(0) : \frac{[\psi^-(1/t)]^d \omega_d}{(2\pi)^d d} \leq e^2 \Gamma(d/\alpha + 1) (\pi^2/\underline{c})^{d/\alpha+1}, \quad t > 0, \quad t\psi^*(\underline{\theta}) < 1/\pi^2. \quad (24)$$

(We may interchange $p_t(0)$ and $[\psi^-(1/t)]^d$ in formulas.)

Also, if $\alpha > 0$ and $\psi \in \text{WLSC}(\alpha, 0, \underline{c})$, then for $t > 0$, $x \in \mathbb{R}^d$,

$$p_t(x) \leq C(d, \alpha) \underline{c}^{-d/\alpha-1} [\psi^-(1/t)]^d. \quad (25)$$

We note in passing that the same argument covers the Gaussian case $\psi(\xi) = |\xi|^2$ and more general exponents otherwise excluded from our general considerations. We also note that analogues of (25) are often obtained by using Nash inequalities [3, 8, 2].

For the α -stable process addressed in (4), by using directly Lemma 16 with $K = 1$ we get

$$p_t(x) \leq \frac{2e^2 \Gamma(\frac{d}{\alpha})}{2^d \pi^{d/2} \alpha \Gamma(d/2)} t^{-d/\alpha},$$

and the constant is not too far off from the exact estimate, obtained by direct integration;

$$\begin{aligned} p_t(x) &\leq p_t(0) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-t|\xi|^\alpha} d\xi = (2\pi)^{-d} \int_0^\infty \omega_d r^{d-1} e^{-tr^\alpha} dr \\ &= (2\pi)^{-d} \omega_d \int_0^\infty s^{d/\alpha-1} \frac{1}{\alpha} t^{-d/\alpha} e^{-s} ds = \frac{2\Gamma(\frac{d}{\alpha})}{2^d \pi^{d/2} \alpha \Gamma(d/2)} t^{-d/\alpha}. \end{aligned} \quad (26)$$

Note that in this case $\psi^-(u) = u^{1/\alpha}$. The above change of variables $tr^\alpha = s$ nicely counterpoints the proof of Lemma 16 and the role of the quantity $t\psi^*(1/|x|)$ appearing, e.g., in Lemma 17.

Corollary 20. *If $\psi \in \text{WLSC}(\underline{\alpha}, \underline{\theta}, \underline{c})$ with $\underline{\theta} > 0$, and $0 < T < \infty$, then $p_t(x) \leq C [\psi^-(1/t)]^d$ for all $0 < t < T$ and $x \in \mathbb{R}^d$, with $C = C(d, T, \underline{\alpha}, \underline{\theta}, \underline{c}, \psi)$.*

Proof. Note that ψ^* satisfies $\text{WLSC}(\underline{\alpha}, \underline{\theta}, \underline{c}/\pi^2)$. Since $\psi^*(0) = 0$ and ψ^* is continuous and unbounded, there is $\theta_1 > 0$ such that $\psi^*(\theta_1) = 1/(\pi^2 T)$. By Lemma 12, ψ^* satisfies $\text{WLSC}(\underline{\alpha}, \underline{\theta} \wedge \theta_1, c_1)$, with a constant c_1 . An application of Proposition 19 completes the proof. In fact, if $\psi \in \text{WLSC}(\underline{\alpha}, 0, \underline{c})$, then $C = C(d, T, \underline{\alpha}, \underline{\theta}, \underline{c}, \psi(\underline{\theta}))$. Indeed, if $\theta_1 < \underline{\theta}$, then $1/(\pi^2 T) = \psi^*(\theta_1) \leq \psi(\underline{\theta})(\underline{c}/\pi^2)^{-1}(\underline{\theta}/\theta_1)^{-\alpha}$, hence $\theta_1^\alpha \geq \underline{c}\theta^\alpha/[\pi^4 T\psi(\underline{\theta})]$, leading to $C = C(T, \underline{\alpha}, \underline{\theta}, \underline{c}, \psi(\underline{\theta}))$. \square

Thus, (25) holds for all $t > 0$, even if $\underline{\theta} > 0$, but the constant deteriorates for large t .

The following main result of our paper gives *common* bounds for unimodal convolution semigroups with scaling. Notably, our second main result, Theorem 26 below, shows in addition that scaling is equivalent to *common* bounds.

Theorem 21. *If $\psi \in \text{WLSC}(\underline{\alpha}, \theta, \underline{c})$, then there is $C^* = C^*(d, \underline{\alpha})$ such that*

$$p_t(x) \leq C^* \min \left\{ (\underline{c})^{-d/\alpha-1} [\psi^-(1/t)]^d, \frac{t\psi^*(1/|x|)}{|x|^d} \right\} \quad \text{if } t > 0 \text{ and } t\psi^*(\theta) < 1/\pi^2.$$

If $\psi \in \text{WLSC}(\underline{\alpha}, \theta, \underline{c}) \cap \text{WUSC}(\overline{\alpha}, \theta, \overline{C})$, then $c^ = c^*(d, \underline{\alpha}, \underline{c}, \overline{\alpha}, \overline{C})$, $r_0 = r_0(d, \underline{\alpha}, \underline{c}, \overline{\alpha}, \overline{C})$ exist with*

$$p_t(x) \geq c^* \min \left\{ [\psi^-(1/t)]^d, \frac{t\psi^*(1/|x|)}{|x|^d} \right\} \quad \text{if } t > 0, \quad t\psi^*(\theta/r_0) < 1 \quad \text{and} \quad |x| < r_0/\theta.$$

Proof. Let $t > 0$. For $x = 0$ the term $t\psi^*(1/|x|)/|x|^d$ in the statement should be ignored—the bounds are to be understood as (22) and (23). Accordingly, below we let $x \in \mathbb{R}^d \setminus \{0\}$. The upper bound now follows from Corollary 7 and Proposition 19.

To prove the lower bound, we consider $\kappa \geq 2$, to be determined later. We have

$$p_t(x) \geq \frac{\mathbb{P}(|x| \leq |X_t| < \kappa|x|)}{|B_{\kappa|x|} \setminus B_{|x|}|} = \frac{d}{\omega_d(\kappa^d - 1)} |x|^{-d} (\mathbb{P}(|X_t| \geq |x|) - \mathbb{P}(|X_t| \geq \kappa|x|)).$$

Let $|x| < \sqrt{a}/\theta$, with $a = a(d, \underline{\alpha}, \overline{C})$ from Lemma 14. We now suppose that $t\psi^*(1/|x|) \leq 1$. By concavity, $s/2 \leq 1 - e^{-s} \leq s$ for $0 \leq s \leq 1$. By Lemma 14 and Corollary 6,

$$\begin{aligned} \mathbb{P}(|X_t| \geq |x|) - \mathbb{P}(|X_t| \geq \kappa|x|) &\geq \frac{a}{2} t\psi^*(1/|x|) - \frac{2e}{e-1} (2d+1) t\psi^*(1/|\kappa x|) \\ &\geq \frac{a}{2} t\psi^*(1/|x|) \left(1 - \frac{4e(2d+1)}{a(e-1)} \frac{\psi^*(1/|\kappa x|)}{\psi^*(1/|x|)} \right). \end{aligned}$$

Recall that $\psi^* \in \text{WLSC}(\underline{\alpha}, \theta, \underline{c}/\pi^2)$. If $|x| < 1/(\kappa\theta)$, then

$$\frac{4e(2d+1)}{a(e-1)} \frac{\psi^*(1/|\kappa x|)}{\psi^*(1/|x|)} \leq \frac{4\pi^2 e(2d+1)}{\underline{c}a(e-1)} \kappa^{-\underline{\alpha}} = \frac{1}{2},$$

where

$$\kappa = (8\pi^2 e(2d+1)/(\underline{c}a(e-1)))^{1/\underline{\alpha}}. \quad (27)$$

Recall that $\underline{c}, a \in (0, 1]$ (see the proof of Lemma 14), so $\kappa \geq 2$, as required. Also, $\kappa^{-1} < \sqrt{a}$. Therefore this choice of κ yields

$$\begin{aligned} p_t(x) &\geq \frac{d}{\omega_d(\kappa^d - 1)} \frac{a}{4} t\psi^*(1/|x|) |x|^{-d} \\ &\geq \frac{ad}{4\omega_d(\kappa^d - 1)} \min \left\{ [\psi^-(1/t)]^d, \frac{t\psi^*(1/|x|)}{|x|^d} \right\}. \end{aligned} \quad (28)$$

We are in a position to verify that the lower bound in the statement of the theorem holds with $r_0 = \min\{\kappa^{-1}, \sqrt{a}\} = \kappa^{-1}$. We thus assume that $t > 0$, $t\psi^*(\theta/r_0) < 1$, and because of the preceding discussion we only need to resolve the case $0 < |x| < r_0/\theta$, $t\psi^*(1/|x|) \geq 1$. By continuity, there is $x^* \in \mathbb{R}^d$ such that $|x| \leq |x^*| < r_0/\theta$ and $t\psi^*(1/|x^*|) = 1$. By (28),

$$\begin{aligned} p_t(x) &\geq p_t(x^*) \geq \frac{ad}{4\omega_d \kappa^d} \frac{1}{|x^*|^d} \geq \frac{ad}{4\omega_d \kappa^d} [\psi^-(1/t)]^d \\ &\geq \frac{ad}{4\omega_d \kappa^d} \min \left\{ [\psi^-(1/t)]^d, \frac{t\psi^*(1/|x|)}{|x|^d} \right\}. \end{aligned}$$

This ends the proof and we may take $c^* = ad\kappa^{-d}/(4\omega_d)$. \square

Remark 5. For the record we note that the constants in the lower bound depend on $d, \underline{\alpha}, \underline{c}, \overline{\alpha}, \overline{C}$ via (27) and Lemma 14, e.g. $c^* = c(d, \underline{\alpha}, \overline{\alpha}) \underline{c}^{d/\underline{\alpha}} \overline{C}^{(\overline{\alpha}-2)(d+\underline{\alpha})/(2\underline{\alpha})}$ and $r_0 = c(\underline{\alpha}, \overline{\alpha}, d) \left(\overline{C}^{\frac{\overline{\alpha}-2}{2}} \underline{c} \right)^{1/\underline{\alpha}}$.

In view of Lemma 17, the two factors in the minima in the statement of Theorem 21 should be interpreted as the approximations of $p_t(x)$ in large time (on-diagonal regime) and small time (off-diagonal regime), correspondingly.

We emphasize that the upper bound in Theorem 21 only requires the lower scaling. For instance the upper bound holds for the 2-regularly varying characteristic exponent $\psi(\xi) =$

$|\xi|^2/[\log(1 + |\xi|^2)]^\beta$ with $\beta \in (0, 1)$, which is in agreement with the outcome of the Davies' method in this case [22].

Tracking constants for the isotropic α -stable Lévy process addressed in (4), we obtain

$$p_t(x) \geq c_\alpha \left(t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}} \right),$$

where

$$c_\alpha = \frac{(2 - \alpha)^{\frac{2(d+\alpha)}{\alpha(2-\alpha)}} (1 - e^{-1})^{\frac{d}{\alpha}} \Gamma\left(\frac{d}{2}, \frac{1}{4}\right)^{\frac{(4-\alpha)(d+\alpha)}{\alpha(2-\alpha)}}}{2^{3+\frac{3d}{\alpha}+\frac{(8-\alpha)(d+\alpha)}{\alpha(2-\alpha)}} \pi^{\frac{d}{2}+\frac{2d}{\alpha}+\frac{(12-2\alpha)(d+\alpha)}{\alpha(2-\alpha)}} (2d+1)^{\frac{d}{\alpha}+\frac{2(d+\alpha)}{\alpha(2-\alpha)}} \Gamma\left(\frac{d}{2}\right)^{\frac{(4-\alpha)(d+\alpha)}{\alpha(2-\alpha)}-1}}. \quad (29)$$

We now list a number of general consequences of Theorem 21. We first complement (16) by a similar lower bound resulting from Theorem 21.

Corollary 22. *If $\psi \in WLSC(\underline{\alpha}, \theta, \underline{c}) \cap WUSC(\overline{\alpha}, \theta, \overline{C})$ and $|x| < r_0/\theta$, then $\nu(x) \geq c^* \psi^*(1/|x|)|x|^{-d}$.*

Corollary 23. *If $\psi \in WLSC(\underline{\alpha}, 0, \underline{c}) \cap WUSC(\overline{\alpha}, 0, \overline{C})$, then (2) holds for all $t > 0$ and $x \in \mathbb{R}^d$.*

Proof. We use (16) and Corollary 22 to obtain

$$\nu(x) \approx \frac{\psi^*(|x|^{-1})}{|x|^d}, \quad x \in \mathbb{R}^d \setminus \{0\}. \quad (30)$$

We then appeal to (22), (25) and Theorem 21. □

By scaling, in particular by Lemma 18, we obtain the following important doubling property, cf. [17] in this connection.

Corollary 24. *If ψ satisfies (global) $WLSC(\underline{\alpha}, 0, \underline{c})$ and $WUSC(\overline{\alpha}, 0, \overline{C})$, then*

$$p_t(2x) \approx p_t(x) \quad \text{and} \quad p_{2t}(x) \approx p_t(x), \quad t > 0, x \in \mathbb{R}^d.$$

Thus, if $\theta = 0$ in Theorem 21, then the global asymptotics of $p_t(x)$ is fully and conveniently reflected by ψ . If $\theta > 0$, then our bounds are only guaranteed to hold in bounded time and space (bounded time for the upper bound). For large times we merely offer the following simple exercise of monotonicity.

Corollary 25. *If $\psi \in WLSC(\underline{\alpha}, \underline{\theta}, \underline{c})$, $0 < |x| < (\pi^{-4}\underline{c})^{1/\underline{\alpha}}/\underline{\theta}$ and $t\psi^*(|x|^{-1}) \geq 1$, then*

$$p_t(0) \leq C(d, \underline{\alpha}) \underline{c}^{-d/\underline{\alpha}-1} t \psi^*(|x|^{-1}) / |x|^d.$$

Proof. Define (threshold time) $t_0 = 1/\psi^*(|x|^{-1})$. By Proposition 2, $\psi^* \in WLSC(\underline{\alpha}, \underline{\theta}, \underline{c}/\pi^2)$, thus

$$t_0 \psi^*(\underline{\theta}) = \frac{\psi^*(\underline{\theta})}{\psi^*(|x|^{-1})} < \pi^{-2}.$$

Since $t \rightarrow p_t(0)$ is decreasing, by Proposition 19 we have for $t \geq t_0$,

$$p_t(0) \leq p_{t_0}(0) \leq C \underline{c}^{-d/\underline{\alpha}-1} [\psi^-(\psi^*(|x|^{-1}))]^d \leq C \underline{c}^{-d/\underline{\alpha}-1} \frac{1}{|x|^d} \leq C \underline{c}^{-d/\underline{\alpha}-1} \frac{t \psi^*(|x|^{-1})}{|x|^d},$$

which completes the proof. □

The next theorem proves that our definitions quite capture the subject of the study.

Theorem 26. Let X_t be an isotropic unimodal Lévy process in \mathbb{R}^d with transition density p , Lévy-Khintchine exponent ψ and Lévy measure density ν . The following are equivalent:

(i) WLSC and WUSC [global WLSC and WUSC] hold for ψ .

(ii) For some $r_0 \in (0, \infty)$ [$r_0 = \infty$] and a constant c ,

$$p_t(x) \geq c \frac{t\psi^*(|x|^{-1})}{|x|^d}, \quad 0 < |x| < r_0, \quad 0 < t\psi^*(|x|^{-1}) < 1.$$

(iii) For some $r_0 \in (0, \infty)$ [$r_0 = \infty$] and a constant c ,

$$\nu(x) \geq c \frac{\psi^*(|x|^{-1})}{|x|^d}, \quad 0 < |x| < r_0.$$

Proof. Theorem 21 and Lemma 17 yield the implication (i) \Rightarrow (ii). The implication (ii) \Rightarrow (iii) follows because $\lim_{t \rightarrow 0^+} p(t, x)/t = \nu(x)$ vaguely on $\mathbb{R}^d \setminus \{0\}$. To prove that (iii) implies (i), we assume that (iii) holds. By the Lévy-Khintchine formula, $\psi(x) = \sigma|x|^2 + \int_{\mathbb{R}^d} (1 - \cos\langle \xi, x \rangle) \nu(dx)$, where $\sigma \geq 0$. Actually, we must have $\sigma = 0$, because

$$\infty > \int_{B_1} |x|^2 \nu(x) dx \geq \int_{B_{1 \wedge r_0}} c|x|^2 \frac{\psi^*(|x|^{-1})}{|x|^d} dx \geq \sigma \int_{B_{1 \wedge r_0}} \frac{c}{|x|^d} dx.$$

By [26, proof of Theorem 6.2] and (5), the following defines a *complete Bernstein function*,

$$\varphi(\lambda) = \int_0^\infty \frac{\lambda}{\lambda + s} s^{-1} \nu(s^{-1/2}) s^{-d/2} ds = \int_0^\infty (1 - e^{-\lambda u}) \mu(u) du, \quad \lambda \geq 0,$$

where $\mu(u) = \mathcal{L}[\nu(s^{-1/2}) s^{-d/2}](u)$. In fact, by changing variables, and (6) for $\lambda > 0$ we have

$$\varphi(\lambda) = 2 \int_0^\infty \frac{\lambda u^2}{\lambda u^2 + 1} \nu(u) u^{d-1} du \approx \int_0^\infty [1 \wedge (\lambda u^2)] \nu(u) u^{d-1} du = \omega_d^{-1} h(\lambda^{-1/2}).$$

By Corollary 3, , there exists $c_1 = c_1(d)$ such that

$$c_1 \varphi(\lambda) \leq \psi(\sqrt{\lambda}) \leq c_1^{-1} \varphi(\lambda), \quad \lambda \geq 0. \quad (31)$$

Since $\varphi'(\lambda) = \int_0^\infty u e^{-\lambda u} \mu(u) du$ and μ is decreasing, Lemma 5 with $n = 0$ and $m = 1$ yields

$$\mu(u) \leq \frac{1}{\gamma(2, 1)} \frac{\varphi'(u^{-1})}{u^2}, \quad u > 0. \quad (32)$$

Using the upper incomplete gamma function and monotonicity of ν , we obtain

$$\nu(x) \leq \frac{\int_{|x|^{-2}}^\infty e^{-s|x|^2} \nu(s^{-1/2}) s^{-d/2} ds}{\int_{|x|^{-2}}^\infty e^{-s|x|^2} s^{-d/2} ds} \leq \frac{\mu(|x|^2)}{|x|^{d-2} \Gamma(1 - d/2, 1)}, \quad x \neq 0. \quad (33)$$

We leave it at that for a moment, to make another observation. As φ is a complete Bernstein function, we have that

$$\varphi_1(\lambda) = \lambda / \varphi(\lambda)$$

is a special Bernstein function (see [26, Definition 10.1 and Proposition 7.1]). Since X_t is pure-jump, $\lim_{|\xi| \rightarrow \infty} \psi(\xi)/|\xi|^2 = 0$. Thus, $\lim_{\lambda \rightarrow \infty} \varphi_1(\lambda) = \infty$. Also, $\varphi(0) = 0$, and by [26, (10.9) and Theorem 10.3], the potential measure of the subordinator with the Laplace exponent φ_1 is absolutely continuous with the density function,

$$f(s) = \int_s^\infty \mu(u) du.$$

In particular $\mathcal{L}f = 1/\varphi_1$. Let $x \in \mathbb{R}^d$ be such that $0 < |x| < r_0$. By (31), (iii), (33) and (32),

$$c_1 c \frac{\varphi(|x|^{-2})}{|x|^d} \leq c \frac{\psi(|x|^{-1})}{|x|^d} \leq \nu(x) \leq \frac{\mu(|x|^2)}{|x|^{d-2}\Gamma(1-d/2, 1)} \leq \frac{\varphi'(|x|^{-2})}{|x|^{d+2}\Gamma(1-d/2, 1)\gamma(2, 1)}. \quad (34)$$

Therefore $c_2\varphi(\lambda) \leq \lambda\varphi'(\lambda)$ for $\lambda > 1/\sqrt{r_0}$, where $c_2 = cc_1\gamma(2, 1)\Gamma(1-d/2, 1)$. This implies that the function $\lambda^{-c_2}\varphi(\lambda)$ is nondecreasing on $(1/\sqrt{r_0}, \infty)$. By Remark 11, and (31), $\varphi \in \text{WLSC}(c_2, 1/\sqrt{r_0}, 1)$. Hence, $\psi \in \text{WLSC}(2c_2, r_0^{-1}, c_1)$.

We now prove the upper scaling. By concavity of φ , $u\varphi'(u) \leq \varphi(u)$. For $0 < s < \sqrt{r_0}$, by (34),

$$f(s) \geq c_3 \int_s^{\sqrt{r_0}} \varphi(u^{-1})u^{-1} du \geq c_3 \int_s^{\sqrt{r_0}} \varphi'(u^{-1})u^{-2} du = c_3 \left(\varphi(s^{-1}) - \varphi(r_0^{-1/2}) \right).$$

Note that φ is strictly increasing. Therefore, for $0 < s < \sqrt{r_0}/2$, we get

$$f(s) \geq c_4\varphi(s^{-1}) = c_4/[s\varphi_1(s^{-1})]. \quad (35)$$

Since f is decreasing and $\mathcal{L}f(u) = 1/\varphi_1(u)$, by Lemma 5 we obtain

$$f(s) \leq \frac{1}{\gamma(2, 1)s^2} \left(\frac{1}{\varphi_1} \right)'(s^{-1}) = \frac{1}{\gamma(2, 1)s^2} \frac{\varphi'_1(s^{-1})}{\varphi_1^2(s^{-1})}.$$

Thus, by (35), $c_5\varphi_1(\lambda) \leq \lambda\varphi'_1(\lambda)$, where $\lambda > 2/\sqrt{r_0}$ and $c_5 = c_4\gamma(2, 1)$. It follows as above that $\varphi_1 \in \text{WLSC}(c_5, 2r_0^{-1/2}, 1)$. Since φ_1 is concave, $\lambda\varphi'_1(\lambda) \leq \varphi_1(\lambda)$, hence $c_5 < 1$. Thus, $\varphi \in \text{WUSC}(1 - c_5, 2r_0^{-1/2}, 1)$. This implies $\psi \in \text{WUSC}(2(1 - c_5), 4r_0^{-1}, c_1^{-2})$. \square

The reader may consult, e.g., example 3 in Section 4.1 for typical asymptotics of $p_t(x)$ and $\nu(x)$. Our next observation results from Corollary 22, (30), (31) and Theorem 26.

Corollary 27. *If the characteristic exponent of a unimodal (isotropic) Lévy process X satisfies global WLSC and WUSC (with exponents $0 < \underline{\alpha} \leq \bar{\alpha} < 2$), then there is a complete subordinate Brownian motion with comparable characteristic exponent, Lévy measure and transition density.*

To indicate an application of Corollary 27, we remark that the boundary Harnack principle proved in [19] for complete subordinate Brownian motions under global scaling conditions should now extend to general subordinate Brownian motions with global scaling.

We see from Corollary 27 that the asymptotics of the characteristic exponent, Lévy measure and transition density observed for complete subordinate Brownian motions are representative among all unimodal Lévy processes with lower and upper global scaling.

We close our discussion with a related result, which negotiates the asymptotics of the Lévy density (at zero) and the Lévy-Khintchine exponent (at infinity) under *approximate* unimodality and weak local scaling conditions. We hope the result will help extend the *common bounds*.

Recall our conventions: $0 < \underline{\alpha}, \bar{\alpha} < 2$, $0 < \underline{c} \leq 1 \leq \bar{C} < \infty$.

Proposition 28. *Let X be a pure-jump symmetric Lévy process with Lévy measure $\nu(dx) = \nu(x)dx$ and characteristic exponent ψ . Suppose that $\theta \in [0, \infty)$, constant $c \in (0, 1]$ and nondecreasing function $f : (0, \infty) \rightarrow (0, \infty)$ exist such that*

$$c \frac{f(1/|x|)}{|x|^d} \leq \nu(x) \leq c^{-1} \frac{f(1/|x|)}{|x|^d}, \quad 0 < |x| < 1/\theta.$$

If $f \in \text{WLSC}(\underline{\alpha}, \theta, \underline{c}) \cap \text{WUSC}(\overline{\alpha}, \theta, \overline{C})$, then $f(|\xi|)$ and $\psi(\xi)$ are comparable for $|\xi| > \theta$. In fact, there is a complete subordinate Brownian motion whose characteristic exponent is comparable to $\psi(x)$ for $|\xi| > \theta$, and whose Lévy measure is comparable to ν on $B_{1/\theta} \setminus \{0\}$.

Proof. Let Y be the pure-jump unimodal Lévy process with Lévy density

$$\nu^Y(x) = f(1/|x|)|x|^{-d} \mathbf{1}_{B_{1/\theta}}(x), \quad x \in \mathbb{R}^d \setminus \{0\}.$$

Let ψ^Y be the characteristic exponent of Y . Using $C = C(d)$ of (16) and Proposition 2, we get

$$f(|\xi|) \leq \pi^2 C \psi^Y(\xi), \quad \xi \in \mathbb{R}^d \setminus \{0\}. \quad (36)$$

On the other hand Corollary 3 yields $C' = C'(d)$ such that

$$\psi^Y(\xi) \leq C' \int_{B_{1/\theta}} [(|\xi||z|)^2 \wedge 1] \frac{f(1/|z|)}{|z|^d} dz, \quad \xi \in \mathbb{R}^d.$$

By scaling of f , for $|\xi| > \theta$ we have

$$\begin{aligned} \psi^Y(\xi) &\leq C' f(|\xi|) \left(\overline{C} |\xi|^{2-\overline{\alpha}} \int_{B_{1/|\xi|}} \frac{dz}{|z|^{d+\overline{\alpha}-2}} + \underline{c}^{-1} |\xi|^{-\underline{\alpha}} \int_{(B_{1/|\xi|})^c} \frac{dz}{|z|^{d+\underline{\alpha}}} \right) \\ &= C' \omega_d \left(\frac{\overline{C}}{2-\overline{\alpha}} + \frac{1}{\underline{c}\underline{\alpha}} \right) f(|\xi|). \end{aligned} \quad (37)$$

By the symmetry $\nu(z) = \nu(-z)$,

$$\psi(\xi) = \int_{\mathbb{R}^d} (1 - \cos\langle \xi, z \rangle) \nu(z) dz \approx \psi^Y(\xi) + \int_{(B_{1/\theta})^c} (1 - \cos\langle \xi, z \rangle) \nu(z) dz, \quad (38)$$

in particular $\psi \approx \psi^Y$ on \mathbb{R}^d if $\theta = 0$. If $\theta > 0$, then the last integral in (38) is bounded by $2\nu((B_{1/\theta})^c) < \infty$; by Proposition 2 we have,

$$\psi^Y(\xi) \geq \psi^Y(\theta)/\pi^2 > 0, \quad |x| > \theta,$$

and so $\psi(\xi) \approx \psi^Y(\xi) \approx f(|\xi|)$ for $|\xi| > \theta$, where in the second comparison we used (36) and (37). It follows that $\psi^Y \in \text{WLSC}(\underline{\alpha}, \theta, \underline{c}_1) \cap \text{WUSC}(\overline{\alpha}, \theta, \overline{C}_1)$. By (31) in the proof of Theorem 26 there is a complete subordinate Brownian motion Z whose characteristic exponent ψ^Z is comparable with ψ^Y on \mathbb{R}^d . Its Lévy density ν^Z is comparable with ν^Y on $B_{r_0/\theta} \setminus \{0\}$ by (16) and Corollary 22, where $r_0 = r_0(d, \underline{\alpha}, \underline{c}_1, \overline{\alpha}, \overline{C}_1)$. The comparability of $\nu^Z(x)$ and $\nu^Y(x)$ also takes place on $B_{1/\theta} \setminus B_{r_0/\theta}$ because the functions are bounded from above and below on the set, as follows from (21) and monotonicity of $f > 0$. Thus, ν and ν^Z are comparable on $B_{1/\theta}$. \square

To clarify, the semigroup of the process X in Proposition 28 is not necessarily unimodal, hence its estimates by f call for other methods, e.g. those based on ν , mentioned in Section 1.

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